$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$C = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

a) $\varphi\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ -x_2 + 3x_3 \\ 2x_1 \end{pmatrix}$

$M_B(\varphi)$:

$\varphi\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \varphi\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad \varphi\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$

$\Rightarrow M_B(\varphi) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 3 \\ 2 & 0 & 0 \end{pmatrix}$

$M_C^B(\psi)$:

$\psi\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 + 2x_3 \\ 2x_1 + x_2 - x_3 \\ 2x_1 + x_2 \\ 2x_1 + 3x_3 \end{pmatrix}$

$\psi\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \quad \psi\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \psi\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}$

$\Rightarrow M_C^B(\psi) = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 1 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$

$\Rightarrow M_C^B(\psi \circ \varphi) = M_C^B(\psi) \cdot M_B^B(\varphi) = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 1 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 3 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -2 & 0 \\ 0 & -3 & 3 \\ 2 & -3 & 3 \\ 8 & -2 & 0 \end{pmatrix}$

b) $T_{B}^{B'} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$T_{B'}^{B} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \underline{-1} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$T_{C}^{C'} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\qquad T_{C'}^{C} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

c) $M_{B'}^{B'}(\varphi) = T_{B'}^{B} \cdot M_{B}^{B}(\varphi) \cdot T_{B}^{B'}$

$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 3 \\ 2 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} =$

$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 \\ -2 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix}$

$M_{C'}^{B'}(\varphi) = T_{C'}^{C} \cdot M_{C}^{B}(\varphi) \cdot T_{B}^{B'} =$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 2 \\ 2 & 1 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 4 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \\ 2 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \\ 2 & 2 & 5 \end{pmatrix}$$

T2c) $E = \{1, x, x^2\}$

$B = \{\underline{2x+1}, 7x+3, x^2+x+2\}$

$$T_E^B = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_B^E = \left(T_E^B\right)^{-1}$$

$$2x+1 = \underline{1} \cdot 1 + \underline{2} \cdot x + \underline{0} \cdot x^2$$

$$7x+3 = \underline{3} \cdot 1 + \underline{7} \cdot x + \underline{0} \cdot x^2$$

$$x^2+x+2 = \underline{2} \cdot 1 + \underline{1} \cdot x + \underline{1} \cdot x^2$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \cdot(-2) \rightsquigarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \cdot(-3) \rightsquigarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & 7 & -3 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ \boxed{} \cdot 3 \\ \end{matrix} \cdot (-11) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -11 \\ 0 & 1 & 0 & -2 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$T_B^E = \begin{pmatrix} 7 & -3 & -11 \\ -2 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad (T_E^B)^{-1} = T_B^E$$

b) $f = a_0 + a_1 x + a_2 x^2$

zur Basis B

$\Rightarrow$ zur Basis E: $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

$$T_B^E \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -11 \\ -2 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 7a_0 - 3a_1 - 11a_2 \\ -2a_0 + a_1 + 3a_2 \\ a_2 \end{pmatrix}$$

$$(7a_0 - 3a_1 - 11a_2) \cdot (2x+1) + (-2a_0 + a_1 + 3a_2) \cdot (7x+3) + (a_2) \cdot (x^2 + x + 2) =$$

$$\cancel{14a_0 x} - 6a_1 x - \cancel{22a_2 x} + 7a_0 - \cancel{3a_1} - \cancel{11a_2} - \cancel{14a_0 x} + 7a_1 x + \cancel{21a_2 x} - 6a_0 + \cancel{3a_1} + \cancel{9a_2} + a_2 x^2 + \cancel{a_2 x} + \cancel{2a_2} =$$

$$a_1 x + a_0 + a_2 x^2$$