$$\varphi\left(\begin{pmatrix}1\\4\end{pmatrix}\right)=\begin{pmatrix}1\\0\end{pmatrix} ; \varphi\left(\begin{pmatrix}2\\7\end{pmatrix}\right)=\begin{pmatrix}3\\1\end{pmatrix}$$

$$\varphi\left(\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\lambda\cdot\begin{pmatrix}1\\4\end{pmatrix}+\mu\cdot\begin{pmatrix}2\\7\end{pmatrix}$$

$$\left(\begin{array}{cc|c}1&2&x\\4&7&y\end{array}\right)\Big]\cdot(-4) \rightsquigarrow \left(\begin{array}{cc|c}1&2&x\\0&-1&y-4x\end{array}\right)\Big]\cdot 2 \rightsquigarrow \left(\begin{array}{cc|c}1&0&-7x+2y\\0&-1&y-4x\end{array}\right)\, |\cdot(-1) \rightsquigarrow \left(\begin{array}{cc|c}\overset{\lambda}{\downarrow} & \overset{\mu}{\downarrow} & \\ 1&0&-7x+2y\\0&1&4x-y\end{array}\right)$$

$$\Rightarrow \boxed{\begin{matrix}\lambda=-7x+2y\\ \mu=4x-y\end{matrix}}$$

$$\underline{\begin{pmatrix}x\\y\end{pmatrix}=(-7x+2y)\cdot\begin{pmatrix}1\\4\end{pmatrix}+(4x-y)\cdot\begin{pmatrix}2\\7\end{pmatrix}}$$

$$\varphi\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=(-7x+2y)\cdot\varphi\left(\begin{pmatrix}1\\4\end{pmatrix}\right)+(4x-y)\cdot\varphi\left(\begin{pmatrix}2\\7\end{pmatrix}\right)=$$

$$-7x+2y\cdot\begin{pmatrix}1\\0\end{pmatrix}+(4x-y)\cdot\begin{pmatrix}3\\1\end{pmatrix}=$$

$$\begin{pmatrix}-7x+2y\\0\end{pmatrix}+\begin{pmatrix}12x-3y\\4x-y\end{pmatrix}=$$

$$\boxed{\begin{pmatrix}5x-y\\4x-y\end{pmatrix}}$$

b) $\varphi\left(\begin{pmatrix}1\\4\end{pmatrix}\right)=\begin{pmatrix}1\\0\end{pmatrix}$ $\varphi\left(\begin{pmatrix}2\\7\end{pmatrix}\right)=\begin{pmatrix}3\\1\end{pmatrix}$ $\varphi\left(\begin{pmatrix}3\\3\end{pmatrix}\right)=\begin{pmatrix}1\\1\end{pmatrix}$

$$\varphi\left(\begin{pmatrix}3\\3\end{pmatrix}\right)=\begin{pmatrix}5\cdot3-3\\4\cdot3-3\end{pmatrix}=\begin{pmatrix}12\\9\end{pmatrix}\neq\begin{pmatrix}1\\1\end{pmatrix}$$

$\Rightarrow$ φ ist keine lineare Funktion!

T23 a)

$\varphi: V \to V; \quad f \mapsto f(2) \cdot x + f'$

$\varphi(0) = 0(2) \cdot x + 0' = 0 + 0 = 0$

$\varphi(f+g) = (f+g)(2) \cdot x + (f+g)' =$

$(f(2) + g(2)) \cdot x + f' + g' =$

$f(2) \cdot x + g(2) \cdot x + f' + g' =$

$f(2) \cdot x + f' + g(2) \cdot x + g' =$

$\varphi(f) + \varphi(g)$

$\varphi(\lambda f) = (\lambda f)(2) \cdot x + (\lambda f)' =$

$\lambda \cdot f(2) \cdot x + \lambda f' =$

$\lambda \cdot (f(2) \cdot x + f') =$

$\lambda \cdot \varphi(f)$

$\Rightarrow \varphi$ ist lin. Abb.

b) $M_E(\varphi)$

$E = \{\underbrace{1}_{e_1}, \underbrace{x}_{e_2}, \underbrace{x^2}_{e_3}\}$

$M_E = \left(\vec{a} \mid \vec{b} \mid \vec{c} \right)$

$\vec{a} = \varphi(e_1)$ zur Basis E geschrieben.

Für jeden Basisvektor berechnen wir φ.

$f: x \mapsto 1$

$$\varphi(1) = (1)(2)\cdot x + (1)' = 1\cdot x + 0 = x \xrightarrow{\text{zur Basis } E} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\varphi(x) = (x)(2)\cdot x + (x)' = 2\cdot x + 1 = 2x+1 \xrightarrow{\text{zur Basis } E} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$f: x \mapsto x$

$$\varphi(x^2) = (x^2)(2)\cdot x + (x^2)' = 4\cdot x + 2x = 6x \xrightarrow{\text{zur Basis } E} \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$$

$$M_E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$6x^2 + 5x + 2 \rightarrow \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$$

$$M_E \cdot \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 48 \\ 0 \end{pmatrix} \rightarrow 48x+5$$

Sei $B = \{x^2, x, 1\}$

$$M_B = \begin{pmatrix} 0 & 0 & 0 \\ 6 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

T24) K ein Körper und $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in K^{2\times 2}$

A ist invertierbar $\Longleftrightarrow$ $ad-bc\neq 0$

und $A^{-1}=\frac{1}{ad-bc}\cdot\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$

"$\Leftarrow$": Sei $ad-bc\neq 0$

Dann ist $B=\frac{1}{ad-bc}\cdot\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$

$$A\cdot\frac{1}{ad-bc}\cdot\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}=\frac{1}{ad-bc}\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}=$$

$$\frac{1}{ad-bc}\cdot\begin{pmatrix}ad-bc & -ba+ba\\ cd-dc & -bc+ad\end{pmatrix}=\begin{pmatrix}ad-bc & 0\\ 0 & ad-bc\end{pmatrix}\cdot\frac{1}{ad-bc}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

"$\Rightarrow$": 1. Fall: $a\neq 0$: Wenn A invertierbar ist, gilt $\operatorname{rang}(A)=2$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{matrix}\\ \leftarrow\end{matrix}\Big]\cdot -\frac{c}{a}\ \rightsquigarrow\ \begin{pmatrix}a & b\\ 0 & \underline{d-\frac{bc}{a}}\end{pmatrix}$$

$$\underbrace{d\cdot\frac{bc}{a}}_{\cdot a}\neq\underbrace{0}_{\cdot a}\ \Rightarrow\ \underline{ad-bc\neq 0}$$

2. Fall: $a=0$

$$A = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \leadsto \begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$

$c \neq 0$, $b \neq 0$ weil $\operatorname{rang}(A) = 2$

$$\Rightarrow \underbrace{ad}_{=0} - \underbrace{bc}_{\neq 0} \neq 0$$

$$ad - bc = \det(A)$$

$$\begin{matrix} c & 0 \\ 0 & d - \frac{bc}{c} \end{matrix} \Big|$$