T33:

$\varphi(v) \in U \cdot \quad (U = \langle u \rangle)$

$$\varphi(v) = \underbrace{\frac{\langle u, v\rangle}{\langle u, u\rangle}}_{\in \mathbb{R}} \cdot u \quad = a \cdot u \quad \text{mit } a \in \mathbb{R}$$

$\Rightarrow$ weil U Unterraum ist (aufgespannt durch u),

git $\varphi(v) \in U$

$\varphi(v) - v \in U^{\perp}$:

$$\langle \varphi(v) - v, u \rangle \overset{?}{=} 0$$

$$\langle \varphi(v) - v, u\rangle = \langle \frac{\langle u,v\rangle}{\langle u,u\rangle} \cdot u - v, u\rangle \overset{\text{Bil.}}{=} \langle \frac{\langle u,v\rangle}{\langle u,u\rangle} \cdot u, u\rangle + \langle -v, u\rangle =$$

$$\frac{\langle u,v\rangle}{\cancel{\langle u,u\rangle}} \cdot \cancel{\langle u,u\rangle} - \langle v, u\rangle = \langle u,v\rangle - \langle u,v\rangle = 0 \checkmark$$

$\Rightarrow \varphi(\cdot)$ eine orthogonale Projektion auf U ist

T34:

a) $$|u-v| = \left|\begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}\right| = \sqrt{(-2)^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$$

b) $$\langle u, w\rangle = 1$$

$$|u| \cdot |w| \cdot \cos(\alpha) = \sqrt{2} \cdot \sqrt{2} \cos(\alpha) = 2 \cdot \cos(\alpha)$$

$$\alpha = \arccos\left(\frac{\langle u,w\rangle}{|u| \cdot |w|}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} \mathrel{\hat{=}} 60^\circ$$

c) $$U^{\perp} = \{x \in \mathbb{R}^3 \mid \langle x, \lambda u + \mu v\rangle = 0 \ \forall \lambda, \mu \in \mathbb{R}\} =$$

$\{x \in \mathbb{R}^3 \mid \lambda <x, u> + \mu <x, v> = 0 \quad \forall \lambda, \mu \in \mathbb{R}\} =$

$\{x \in \mathbb{R}^3 \mid <x, u> = <x, v> = 0\} =$

$\left\{x \in \mathbb{R}^3 \mid <\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}> = <\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}> = 0\right\} =$

$\text{Kern} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix} \rfloor \cdot -1 = \text{Kern} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \end{pmatrix} = < \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} >$

T35) a) $<u \times v, w> = < \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} > =$

$w_1(u_2 v_3 - u_3 v_2) + w_2(u_3 v_1 - u_1 v_3) + w_3(u_1 v_2 - u_2 v_1) =$

$u_1 v_2 w_3 + v_1 w_2 u_3 + w_1 u_2 v_3 - u_3 v_2 w_1 - v_3 w_2 u_1 - w_3 u_2 v_1 =$

$\det \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} = \det(u|v|w)$

b) Aus a) folgt für $w = u$: $<u \times v, u> = \det(u|v|u) = 0$ ✓

——— " ——— $w = v$: $<u \times v, v> = \det(u|v|v) = 0$ ✓

c) "$\Rightarrow$": Es seien u, v lin. unabh. Dann $\exists w \in \mathbb{R}^3$, sodass $\{u, v, w\}$ Basis von $\mathbb{R}^3$ ist.

$\Rightarrow \det(u|v|w) \neq 0 \overset{(a)}{\Rightarrow} <u \times v, w> \neq 0 \Rightarrow u \times v \neq 0$

"$\Leftarrow$": Es sei $u \times v \neq 0$. $\lambda, \mu \in \mathbb{R}$ mit $\lambda u + \mu v = 0$

$\rightarrow$ Da "$\times$" bilinear ist, gilt

$0 = 0 \times v = (\lambda u + \mu v) \times v = \lambda \cdot u \times v + \mu \cdot \underbrace{v \times v}_{=0} = \lambda \cdot \underbrace{u \times v}_{\neq 0} \Rightarrow \lambda = 0$

$0 = 0 \times u = (\mu v) \times u = \mu \cdot \underbrace{v \times u}_{\neq 0} \Rightarrow \mu = 0$

$\Rightarrow \lambda = \mu = 0 \quad \Rightarrow u, v$ lin. unabh.